Vector Proof – Collinear Points
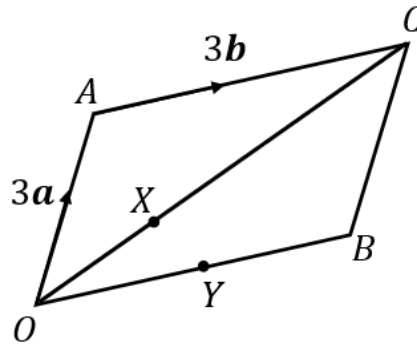
(a)

$OACB$ is a parallelogram. $\overrightarrow{OA} = 3\boldsymbol{a}$ and $\overrightarrow{AC} = 3\boldsymbol{b}$. Y is the midpoint of OB and X divides the line OC in the ratio $1 : 2$. Show that the points A, X and Y are collinear.

$$\overrightarrow{AY} = -3\boldsymbol{a} + \frac{3}{2}\boldsymbol{b}$$

$$\overrightarrow{AX} = -3\boldsymbol{a} + \boldsymbol{a} + \boldsymbol{b} = -2\boldsymbol{a} + \boldsymbol{b} \qquad \overrightarrow{AX} = \frac{2}{3}\overrightarrow{AY}$$

Since $\overrightarrow{AY}$ is a multiple of $\overrightarrow{AX}$ they are parallel, and there is a common point A, the points A, X and Y are collinear.

(b)

$OACB$ is a trapezium. $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{AB} = \boldsymbol{b}$. $\overrightarrow{OC} = 3\overrightarrow{AB}$ and X divides the line OB in the ratio $3 : 1$. Show that the points A, X and C are collinear.

$$\overrightarrow{AC} = -\boldsymbol{a} + 3\boldsymbol{b}$$

$$\overrightarrow{AX} = -\boldsymbol{a} + \frac{3}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}$$

$$\overrightarrow{AX} = -\frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}$$

$$\overrightarrow{AC} = 4\overrightarrow{AX}$$

Since $\overrightarrow{AC}$ is a multiple of $\overrightarrow{AX}$ they are parallel, and there is a common point A, the points A, X and C are collinear.
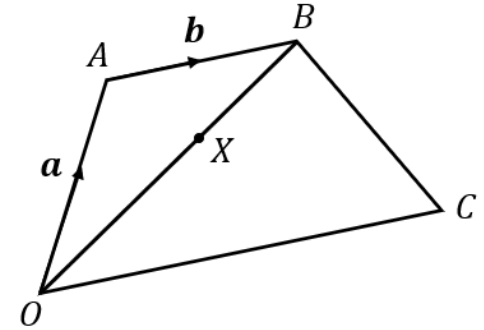
(c)

In the triangle OAB, $\overrightarrow{OX} = \boldsymbol{a}$ and $\overrightarrow{AB} = \boldsymbol{b}$. X is the midpoint of OA and the point Y divides the line AB in the ratio $2 : 1$. $\overrightarrow{OB} = \overrightarrow{BC}$. Show that the points X, Y and C are collinear.

$$\overrightarrow{XC} = 3\boldsymbol{a} + 2\boldsymbol{b}$$

$$\overrightarrow{XY} = \boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$$

$$\overrightarrow{XC} = 3\overrightarrow{XY}$$

Since $\overrightarrow{XC}$ is a multiple of $\overrightarrow{XY}$ they are parallel, and there is a common point X, the points X, Y and C are collinear.
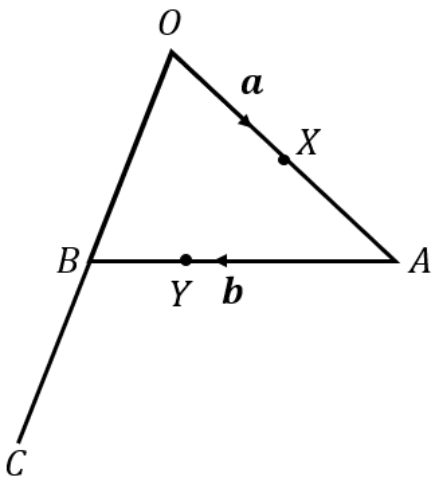
(d)

$\overrightarrow{OA} = 4\boldsymbol{a} - \boldsymbol{b}$, $\overrightarrow{AB} = \boldsymbol{a} + 2\boldsymbol{b}$ and $\overrightarrow{OC} = \boldsymbol{a} + \boldsymbol{b}$. $\overrightarrow{AB} = \overrightarrow{BD}$. The point X divides the line AC in the ratio $6 : 1$. Show that O, X and D are collinear.

$$\overrightarrow{OD} = 6\boldsymbol{a} + 3\boldsymbol{b}$$

$$\overrightarrow{OX} = \frac{10}{7}\boldsymbol{a} + \frac{5}{7}\boldsymbol{b}$$

$$\overrightarrow{OD} = \frac{21}{5}\overrightarrow{OX}$$

Since $\overrightarrow{OD}$ is a multiple of $\overrightarrow{OX}$ they are parallel, and there is a common point O, the points O, X and D are collinear.